Experiment, Outcome and Sample Space

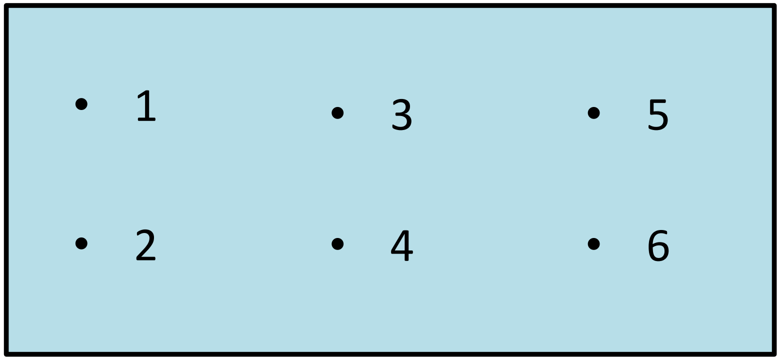
Section 4.1

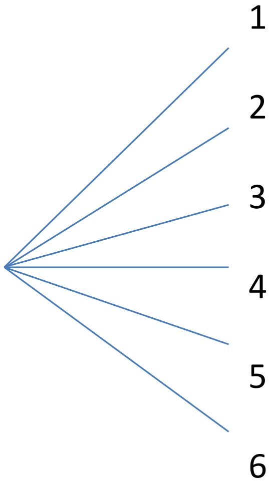
Experiment, Outcome & Sample Space

- Experiment – a process that results in one and only one of many observations.
- Outcomes – observations from an experiment
- Sample Space – collection of all outcomes for an experiment
- Venn Diagrams & Tree Diagrams – pictorial representation of the sample space

Example

- Experiment: Roll a die once
- Outcomes: 1, 2, 3, 4, 5, 6
- Sample Space: {1,2,3,4,5,6}





Event

- An event is a collection of one or more of the outcomes of the experiment.
  - Simple (Elementary) Event – each of the final outcomes for an experiment includes only one outcome, denoted by E_i
  - Compound (Composite) Event – a collection of more than one outcome for an experiment

Example

An ATM is stocked with \$10 and \$20 bills. When a customer withdraws \$40, the machine dispenses either two \$20 bills or four \$10 bills. If 2 customers withdraw \$40 each, how many outcomes are possible? Show all the outcomes in a Venn diagram.

List all the outcomes of the following events and mention which are simple and which are compound.

a. Exactly one customer receives \$20 bills
b. Both customers receive \$10 bills
c. At most one customer receives \$20 bills
d. The 1st customer receives \$10 bills and the 2nd receives \$20 bills.

Calculating Probability

Section 4.2

Properties of Probability

Probability is a numerical measure of the likelihood that a specific event will occur.

Properties of Probability

1. The probability of an event always lies in the range 0 to 1.

$$0 \leq P(E_i) \leq 1$$
$$0 \leq P(A) \leq 1$$

   – Event that cannot occur has zero probability and is called an impossible (or null) event

   – Event that is certain to occur has probability equal to 1 and is called a sure (or certain) event

2. The sum of probabilities of all simple events (or final outcomes) for an experiment, denoted by $\sum P(E_i)$ is always 1.

$$\sum P(E_i) = P(E_1) + P(E_2) + P(E_3) + \cdots = 1$$

Examples of Properties of Probability

The probability that the sun will rise is 1. It is a sure event.

The probability that you will never make a mistake is 0. It is an impossible event.

The probability of rolling a die is

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

Classical Probability

- If 2 or more outcomes have the same probability of occurrence, they are said to be equally likely outcomes.
- Classical Probability Rule
  - Only applies for equally likely outcomes
  - $P(E_i) = \frac{1}{Total\ number\ of\ outcomes\ for\ the\ experiment}$
  - $P(A) = \frac{Number\ of\ outcomes\ favorable\ to\ A}{Total\ number\ of\ outcomes\ for\ the\ experiment}$

Example

- What is the probability that you will draw the ace of hearts from a standard 52-card deck of cards?

$$P(Ace\ of\ Hearts) = \frac{1}{52} = .019$$

- What is the probability that you will draw a face card from a standard 52-card deck of cards?

$$P(Face\ Card) = \frac{\#\ of\ face\ cards}{Total\ \#\ of\ cards} = \frac{12}{52} = .231$$

Relative Frequency Concept of Probability

- If an experiment is repeated n times and an event A is observed f times, then $P(A) = \frac{f}{n}$
- A way to approximate probability for events not equally likely to occur by using the frequency distribution.
- If you are dealing with a population the relative frequency is an exact probability
- Law of Large Numbers – If an experiment is repeated again and again, the probability of an event obtained from the relative frequency approaches the actual or theoretical probability.

Examples of Relative Frequency Probability Cases

- The probability that the next car that comes out of an auto factory is a "lemon"
- The probability that a randomly selected family owns a home
- The probability that a randomly selected woman is an excellent driver
- The probability that an 80-year-old person will live for at least 1 more year
- The probability that a randomly selected adult is in favor of increasing taxes to reduce the national debt
- The probability that a randomly selected person owns a sport-utility vehicle (SUV)

Example

A random sample of 2000 adults showed that 1320 of them have shopped at least once on the Internet. What is the (approximate) probability that a randomly selected adult has shopped on the Internet?

$$P(shopped\ on\ Internet) = \frac{f}{n} = \frac{1320}{2000} = .66$$

Subjective Probability

- Experiments that have neither equally likely outcomes nor the potential of being repeated.
- Probability assigned to an event based on subjective judgment, experience, information, and belief.
- Assigned arbitrarily
- Influenced by biases, preferences, and experience of the person assigning the probability.

Example

What is the probability that you will earn an A in Elementary Statistics?

- Not equally likely outcomes
- No potential to be repeated
- Can use other information to make a decision on the probability but no rigorous way to determine.

Examples of Subjective Probability

- The probability that Joe will lose the lawsuit he has filed against his landlord
- The probability that the New York Giants will win the Super Bowl next season
- The probability that the Dow Jones Industrial Average will be higher at the end of the next trading day

Calculating Probability Examples

- A hat contains 40 marbles. Of them, 18 are red and 22 are green. If one marble is randomly selected out of this hat, what is the probability that this marble is red? Green?

  Answer: P(red) = .45, P(green) = .55

- A die is rolled once. What is the probability that a number less than 5 is obtained? A number 3 to 6?

  Answer: P(<5) = P(1,2,3,4)=.667, P(3,4,5,6) = .667

- In a statistics class of 42 students, 28 have volunteered for community service in the past. Find the probability that a randomly selected student from this class has volunteered for community service in the past.

  Answer: P(volunteered) = .667

- In a group of 50 car owners, 8 own hybrid cars. If one car owner is selected at random from this group, what is the probability that this car owner owns a hybrid car?

  Answer: P(own hybrid) = .16

- Out of the 3000 families who live in a given apartment complex in New York City, 600 paid no income tax last year. What is the probability that a randomly selected family from these 3000 families did pay income tax last year?

  Answer: P(pay tax) = .8

- The television game show *The Price Is Right* has a game called the Shell Game. The game has four shells, and one of these four shells has a ball under it. The contestant chooses one shell. If this shell contains the ball, the contestant wins. If a contestant chooses one shell randomly, what is the probability that a contestant loses? Wins? Do the probabilities add up to 1? Why?

  Answer: P(lose) = .75, P(win) = .25

More Calculating Probability Examples

- There are 1265 eligible voters in a town, and 972 of them are registered to vote. If one eligible voter is selected at random from this town, what is the probability that this voter is registered? Not registered? Do the probabilities add up to 1? Why?

  Answer: P(registered) = .768, P(not registered) = .232
- A sample of 500 large companies showed that 120 of them offer free psychiatric help to their employees who suffer from psychological problems. If one company is selected at random from this sample, what is the probability that this company offers free psychiatric help to its employees who suffer from psychological problems? What is the probability that this company does not offer free psychiatric help to its employees who suffer from psychological problems? Do these two probabilities add up to 1.0? If yes, why?

  Answer: P(free help) = .24, P(no free help) = .76

Marginal Probability, Conditional Probability, and Related Probability Concepts

Section 4.3

Marginal Probability

- Marginal probability is the probability of a single event without consideration of any other event.
  - Also called simple probability

Marginal Probability Example

Five hundred employees were selected from a city’s large private companies, and they were asked whether or not they have any retirement benefits provided by their companies. Based on this information, the following two-way classification table was prepared.

	Have Retirement Benefits	No Retirement Benefits
Men	225	75
Women	150	50

Marginal Probability Example Continued

	Have Retirement Benefits	No Retirement Benefits	Total
Men	225	75	300
Women	150	50	200
Total	375	125	500

- Calculate the marginal probabilities by dividing margins by grand totals.

$$P(man) = \frac{Number\ of\ men}{Total\ number\ of\ employees} = \frac{300}{500} = .6$$

$$P(woman) = \frac{Number\ of\ women}{Total\ number\ of\ employees} = \frac{200}{500} = .4$$

$$P(Retirement) = \frac{\#\ with\ retirement\ benefits}{Total\ number\ of\ employees} = \frac{375}{500} = .75$$

$$P(No\ Retirement) = \frac{\#\ without\ retirement\ benefits}{Total\ number\ of\ employees} = \frac{125}{500} = .25$$

Another Marginal Probability Example

Two thousand randomly selected adults were asked if they are in favor of or against cloning. The following table gives the responses.

	In Favor	Against	No Opinion	Total
Male	395	405	100	900
Female	300	680	120	1100
Total	695	1085	220	2000

$$P(Male) = \frac{900}{2000} = .45$$

$$P(Female) = \frac{1100}{2000} = .55$$

$$P(In\ Favor) = \frac{695}{2000} = .348$$

$$P(Against) = \frac{1085}{2000} = .543$$

$$P(No\ Opinion) = \frac{220}{2000} = .11$$

Conditional Probability

- Conditional probability is the probability that an event will occur *given* that another event has already occurred. If A and B are two events, then the conditional probability of A given B is written as $P(A|B)$ and read as “the probability of A given that B has already occurred.”

Conditional Probability Example

	Have Retirement Benefits	No Retirement Benefits	Total
Men	225	75	300
Women	150	50	200
Total	375	125	500

- Calculate conditional probability by dividing the value in the cell that satisfies both conditions by the total of the given condition.

$$P(retirement|man) = \frac{\#\ of\ men\ with\ retirement\ benefits}{Total\ number\ of\ men} = \frac{225}{300} = .75$$

- NOTE: $P(A|B) \neq P(B|A)$

$$P(man|retirement) = \frac{\#\ of\ men\ with\ retirement\ benefits}{Total\ number\ with\ retirement\ benefits} = \frac{225}{375} = .6$$

Another Conditional Probability Example

	In Favor	Against	No Opinion	Total
Male	395	405	100	900
Female	300	680	120	1100
Total	695	1085	220	2000

$$P(Male|In\ Favor) = \frac{395}{695} = .568$$

$$P(Female|In\ Favor) = \frac{300}{695} = .432$$

$$P(Male|Against) = \frac{405}{1085} = .373$$

$$P(Female|Against) = \frac{680}{1085} = .627$$

$$P(Male|No\ Opinion) = \frac{100}{220} = .455$$

$$P(Female|No\ Opinion) = \frac{120}{220} = .545$$

$$P(In\ Favor|Male) = \frac{395}{900} = .439$$

$$P(In\ Favor|Female) = \frac{300}{1100} = .359$$

$$P(Against|Male) = \frac{405}{900} = .45$$

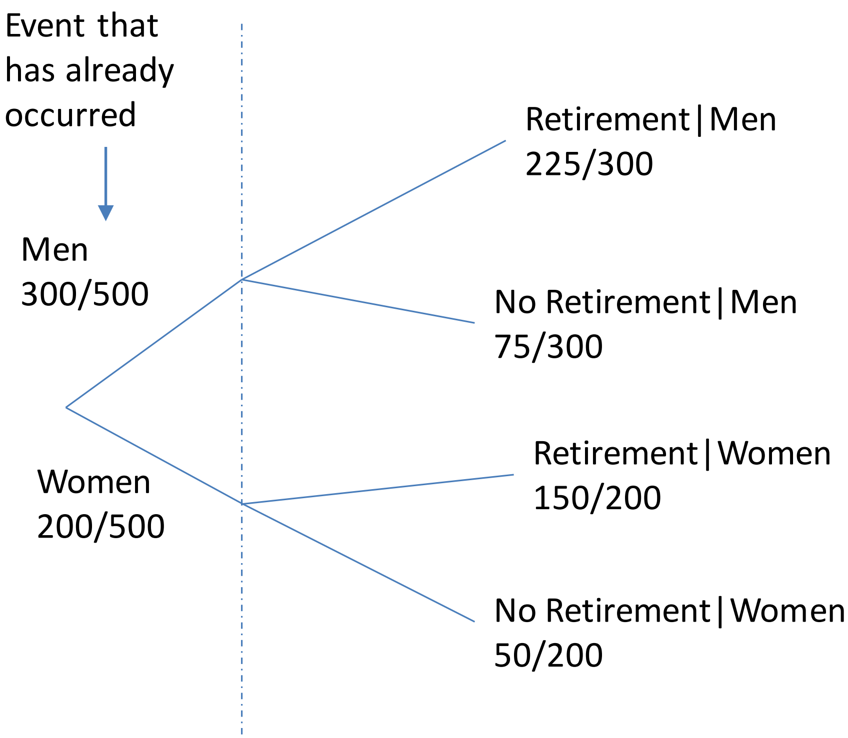
$$P(Against|Female) = \frac{680}{1100} = .618$$

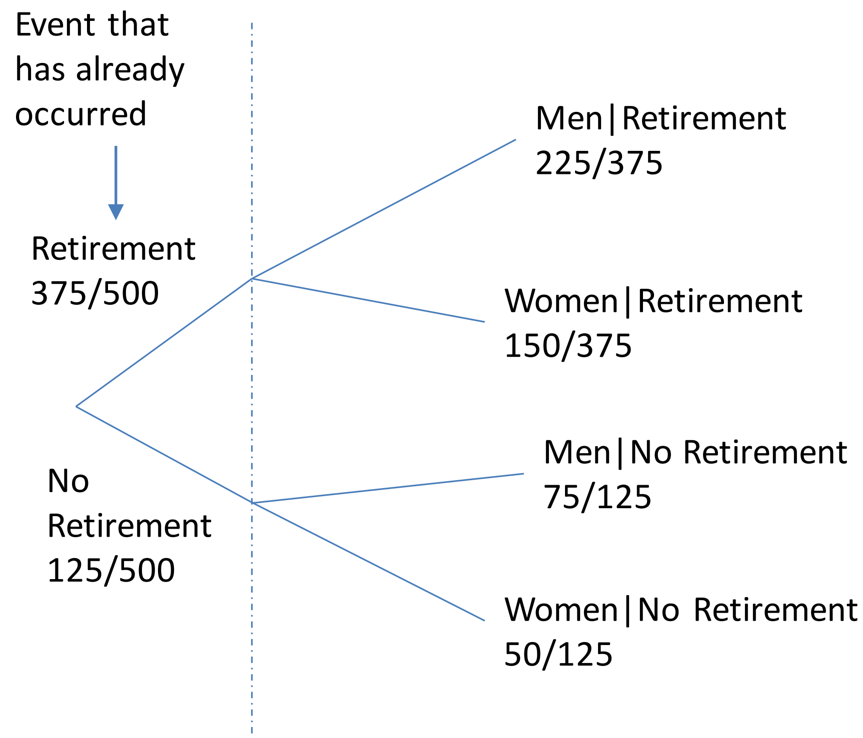
$$P(No\ Opinion|Male) = \frac{100}{900} = .111$$

$$P(No\ Opinion|Female) = \frac{120}{900} = .133$$

Conditional Probability Tree Diagram

	Have Retirement Benefits	No Retirement Benefits	Total
Men	225	75	300
Women	150	50	200
Total	375	125	500





Mutually Exclusive Events

- Events that cannot occur together are called mutually exclusive events
- For any experiment the final outcomes are always mutually exclusive

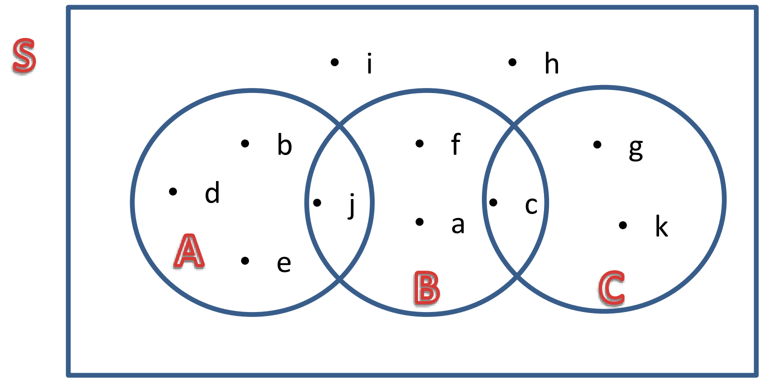
<u>Example</u>: A statistical experiment has 11 equally likely outcomes that and denoted by *a, b, c, d, e, f, g, h, i, j,* and *k.*

Consider the events: A={b, d, e, j}, B={a, c, f, j}, C={c, g, k}

A and B are not mutually exclusive (mutually nonexclusive) – both contain j

A and C are mutually exclusive – no overlap

B and C are mutually nonexclusive – both contain c.



Independent Events

- Two events are independent if the occurrence of one does not affect the probability of the occurrence of the other.
- A and B are independent events if $P(A|B) = P(A)$ or $P(B|A) = P(B)$

Example: A statistical experiment has 11 equally likely outcomes that and denoted by *a, b, c, d, e, f, g, h, i, j,* and *k.*

Consider the events: A={b, d, e, j}, B={a, c, f, j}, C={c, g, k}

$P(A|B) = \frac{1}{4} = .25$

$P(A) = \frac{4}{11} = .364$

so $P(A|B) \neq P(A)$

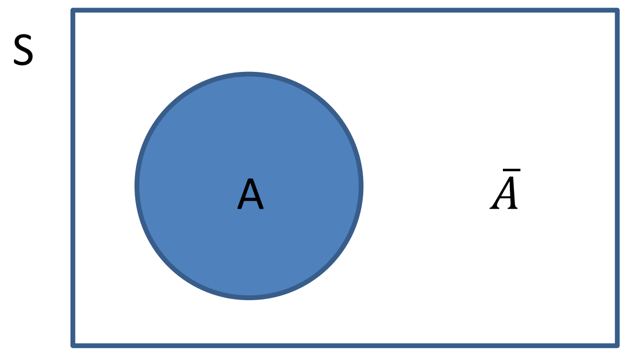
A and B are dependent events

Observations about Mutually Exclusive and Independent/Dependent Events

1. Two events are either mutually exclusive or independent.
   a. Mutually exclusive events are always dependent
   b. Independent events are never mutually exclusive.
2. Dependent events may or may not be mutually exclusive.

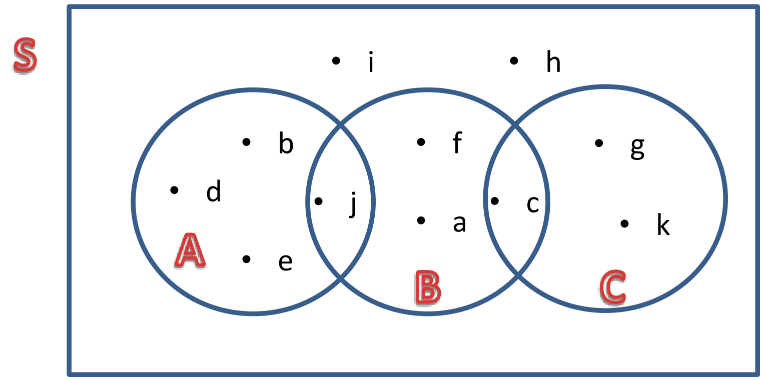
Complementary Events

- Mutually exclusive events that include all outcomes for an experiment
- The complement of event A, denoted by $\bar{A}$ and read as "A bar" or "A complement" is the event that includes all outcomes for an experiment that are not in A.
- $P(A) + P(\bar{A}) = 1$



Complementary Events Example

- $A = \{b, d, e, j\}$
- $\bar{A} = \{a, c, f, g, h, i, k\}$
- $P(A) = \frac{4}{11} = .364$
- $P(\bar{A}) = \frac{7}{11} = .636$
- Note: $P(A) + P(\bar{A}) = 1$



Example

A survey asked people to choose their favorite junk food from a list of choices. The table below represents the results of the 8002 people who responded to the survey.

Favorite Junk Food	Female	Male	Total
Chocolate	1518	531	**2049**
Sugary Candy	218	127	**345**
Ice Cream	685	586	**1271**
Fast Food	312	463	**775**
Cookies	431	219	**650**
Chips	458	649	**1107**
Cake	387	103	**490**
Pizza	792	523	**1315**
Total	**4801**	**3201**	**8002**

If one person is selected at random from this sample of 8002 respondents, find the probability that this person

1. Is a female
2. Responded chips
3. Responded chips given that this person is a female
4. Responded chocolate given that this person is a male

Are the events chips and cake mutually exclusive? What about the events chips and female? Why or why not?

Are the events chips and female independent? Why or why not?

More Examples

1. Let A be the event that a number less than 3 is obtained if we roll a die once. What is the probability of A? What is the complementary event of A, and what is its probability?
   - $A = \{1,2\}$
   - $P(A) = \frac{2}{6} = .333$
   - $\bar{A} = \{3,4,5,6\}$
   - $P(\bar{A}) = 1 - .333 = .667$

2. Thirty percent of last year's graduates from a university received job offers during their last semester in school. What are the two complementary events here and what are their probabilities?
   - $A = graduates\ who\ receive\ a\ job\ offer$
   - $\bar{A} = graduates\ who\ don't\ receive\ a\ job\ offer$
   - $P(A) = .3$
   - $P(\bar{A}) = 1 - .3 = .7$

Intersection of Events and the Multiplication Rule

Section 4.4

Intersection of Events

- Let A and B be two events. The intersection of A and B represents the collection of all outcomes that are common to both A and B.
- Denoted by A and B, $A \cap B$, or AB
- Joint Probability – The probability of the intersection of two events is called their join probability. It is written as P(A and B) or $P(A \cap B)$ or P(AB).
- Multiplication Rule – The probability of the intersection of two events A and B is $P(A\ and\ B) = P(A)P(B|A) = P(B)P(A|B)$

Joint Probability Example

	Have Retirement Benefits	No Retirement Benefits	Total
Men	225	75	300
Women	150	50	200
Total	375	125	500

Find the probability that a randomly selected person was a woman and had retirement benefits.

$$P(woman \,\&\, retirement) = P(woman)P(retirement|woman)$$

$$= \frac{200}{500} \times \frac{150}{200} = \frac{150}{500} = .3$$

*Notice this is the cell value divided by the grand total.

Conditional Probability Using Joint Probability

- If A and B are two events, then $P(B|A) = \frac{P(AB)}{P(A)}$ and $P(A|B) = \frac{P(AB)}{P(B)}$
  $$given\ P(A) \neq 0\ and\ P(B) \neq 0$$

- Example: The probability that a student graduating from Suburban State University has student loans to pay off after graduation is .60. The probability that a student graduating from this university has student loans to pay off after graduation and is male is .24. Find the probability that a randomly selected student from this university is a male given they have student loans to pay off after graduation.
  - $P(M|Loans) = \frac{P(M\ \&\ Loans)}{P(Loans)} = \frac{.24}{.60} = .4$

Multiplication Rule for Independent Events

- The probability of the intersection of two independent events A and B is $P(AB) = P(A)P(B)$
- Example: A and B are independent events. P(A)=.17 and P(B)=.44
  - P(AB)=.17*.44=.075
- The probability that a farmer is in debt is .8. What is the probability that three randomly selected farmers are in debt? Assume independence of events.
  - $P(F_1 \,\&\, F_2 \& F_3) = P(F_1)P(F_2)P(F_3) = .8 \times .8 \times .8$

Joint Probability of Mutually Exclusive Events

- The joint probability of two mutually exclusive events is always zero. If A and B are mutually exclusive events, the P(A and B)=0
- Example: H=event that head is flipped,
  T=event that tails is flipped
  P(H & T) = 0

Union of Events and the Addition Rule

Section 4.5

Union of Events

- Let A and B be two events. The union of events is the collection of all outcomes that belong either to A or to B or to both A and B
- Denoted by (A or B) or $A \cup B$
- Addition Rule for Probability of Union of Events
  - $P(A\ or\ B) = P(A) + P(\ B) - P(A \cap B)$
  - Add marginal probabilities and subtract joint probability

Union Example

Favorite Junk Food	Female	Male	Total
Chocolate	1518	531	**2049**
Sugary Candy	218	127	**345**
Ice Cream	685	586	**1271**
Fast Food	312	463	**775**
Cookies	431	219	**650**
Chips	458	649	**1107**
Cake	387	103	**490**
Pizza	792	523	**1315**
Total	**4801**	**3201**	**8002**

Find the probability that a randomly selected person is male or choose pizza.

$$P(M\ or\ Pizza) = P(M) + P(Pizza) - P(M\ \&\ Pizza)$$

$$= \frac{3201}{8002} + \frac{1315}{8002} - \frac{523}{8002} = \frac{3993}{8002} = .499$$

Addition Rule for Probability of Union of Mutually Exclusive Events

- If A and B are mutually exclusive events,

$$P(A \text{ or } B) = P(A) + P(B)$$

Favorite Junk Food	Female	Male	Total
Chocolate	1518	531	**2049**
Sugary Candy	218	127	**345**
Ice Cream	685	586	**1271**
Fast Food	312	463	**775**
Cookies	431	219	**650**
Chips	458	649	**1107**
Cake	387	103	**490**
Pizza	792	523	**1315**
Total	**4801**	**3201**	**8002**

Find the probability that a randomly selected person responded chocolate or fast food.

Events chocolate and fast food are mutually exclusive, so

$$P(choc \text{ or } fast\ food)$$
$$= P(choc) + P(fast\ food)$$
$$= \frac{2049}{8002} + \frac{775}{8002}$$
$$= .353$$

Counting Rule, Factorials, Combinations & Permutations

Section 4.6

Counting Rule to Find Total Outcomes

- The total number of outcomes for an experiment can be found by multiplying the number of outcomes for each step of the experiment.
- Example
  - Experiment: Roll a die 4 times
  - Total # of outcomes = 6 x 6 x 6 x 6 = 1296

Factorial

- The symbol n!, read as “n factorial” represents the product of all integers from n to 1.

$$n! = n(n-1)(n-2)(n-3)\ldots 3 \times 2 \times 1$$

- Example: 6! = 6 x 5 x 4 x 3 x 2 x 1 = 720

Combinations

- Combinations give the number of ways x elements can be selected from n elements.
  - Denoted ${}_nC_x$ and read "the number of combinations of n elements selected x at a time"
  - Without replacement
  - Order does not matter
- Example: Given 6 letters A,B,C,D,E,F how many different combinations of 2 letters can you make?

AB	AC	AD	AE	AF
BC	BD	BE	BF	
CD	CE	CF		
DE	DF			
EF				

15 different combinations

Calculating the Number of Combinations

- The number of combinations for selecting x of n distinct elements is given by

$$ {}_{n}C_{x} = \frac{n!}{x!\,(n-x)!} $$

- Example:

$$ {}_{6}C_{2} = \frac{6!}{2!(6-2)!} = \frac{6\times5\times4\times3\times2\times1}{(2\times1)(4\times3\times2\times1)} = 15 $$

Permutations

- Permutations give the selections of x elements from n different elements in such a way that order of selection is important
  - Denoted ${}_{n}P_{x}$ and read "the number of permutations of n elements selected x at a time"
  - Without replacement
  - Order matters
- Example: Given 6 letters A,B,C,D,E,F how many different permutations of 2 letters can you make?

AB	AC	AD	AE	AF
BA	BC	BD	BE	BF
CA	CB	CD	CE	CF
DA	DB	DC	DE	DF
EA	EB	EC	ED	EF
FA	FB	FC	FD	FE

30 permutations

Calculating the Number of Permutations

- The number of permutations for selecting x of n distinct elements is given by

$$_{n}P_{x} = \frac{n!}{(n-x)!}$$

- Example:

$$_{6}P_{2} = \frac{6!}{(6-2)!} = \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = 30$$